1.138J/2.062J, WAVE PROPAGATION

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CHAPTER FIVE

Miscellaneous Topics

1 Wave localization in a random medium

There are numerous situations where one needs to know how waves propagate through a medium with random impurities: light through sky with dust particles, sound through water with bubbles, elastic waves through a solid with cracks, fibers, cavities, hard or soft grains. Sea waves over a irregular topography, etc. In these situations several kinds of questions can be of physical interest : determinstic (sinusoidal or impulsive) waves through a random medium, random waves through a deterministically irregular medium, and random waves through a random medium.

There is an extensive literature on the propagation of infinitesimal sinusoidal waves in random media. Based on linearized field equations, perturbation theories have been developed for cases where the random inhomogeneity is weak and the fluctuation length scale is comparable to the typical wave length (see, Chernov, 1960; Keller, 1964, Karal & Keller, 1964; Chen & Soong, 1972). Diagrammatic techniques have also been employed (Frisch, 1968; Elter & Molyneux, 1972). If the inhomogeneities extend over a large spatial region, multiple scattering yields a change in the wavenumber (or phase velocity) as well as an amplitude attenuation over a large distance. These changes amount to a shift of the complex propagation constant with the real part corresponding to the wavenumber and the imaginary part to attenuation. In particular, the spatial attenuation (localization) is a distinctive feature of randomness and is effective for a broad range of incident wave frequencies.This is in sharp contrast to periodic inhomogeneities which cause strong scattering only for certain frequency bands (Bragg scattering, see e.g., Nayfeh). Phillip W. Anderson (1958) was the first to show that the quantum- mechanical motion of a particle in a random potential can be localized in space, turning a conductor to an insulator. This phenomoena, now called *Anderson localization*, is now known to be important in classical mechanical systems too. A survey of localization in many types of classical waves based on linearized theories can be found in the monograph by Sheng (1998). For surface water waves, localization by strong inhomogeneities have also been treated by semi-numerical means for surface water waves over randomly rough seabed where the height of the roughness is comparable to the mean depth (Devillard et al, 1988; Nachbin & Papanicolaou, 1992; Nachbin, 1995). Experimental confirmation has been reported by Belzons et al (1988).

For weak inhomogeneties, the shift of propagation constant amounts to slow spatial modulations with a length scale much longer than the wavelnegth by a factor inversely proportional to the correlation of the fluctuations. In this section we use the *method of multiple scales* to examine weak random inhomogeneities. The simple case of an elastically supported string is used as an example, while extensions to other waves can be anticipated. After deriving the envelope equation, physcial implications will be explored.

We begin with the equation for the lateral displacement of a taut string, which is buried in a linear elastic medium,

$$\rho\frac{\partial^2 V}{\partial t^2} - T\frac{\partial^2 V}{\partial x^2} + K(1+\epsilon M(x))V = 0 \tag{1.1}$$

V denotes the lateral displacement, ρ the mass per unit length, T tension in the string, K the mean spring constant of the surrounding medium, $\epsilon K M(x)$ the random fluctuations of the linear spring force. We assume that M has zero mean and the typical length scale of $O(1/k)$. For the sake of demonstration we have chosen to let the linear part of the spring to contain random irregularities. In principle the randomness can appear in the density ρ also.

Since the correlation of a random function is proportional to the square of the amplitude of random fluctuations, the length scale of modulation due to randomness must be of the order $O(1/k\epsilon^2$. Let us introduce fast and slow variables $x, x_2 = \epsilon^2 x$ and further assume two-vairiable expansions,

$$V = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots, \quad \text{with} \;\; V_n = V_n(x, x_2, t), n = 0, 1, 2, ... \tag{1.2}$$

In the multiplescale formalism, we first pretend the two variables to independent, then use the definition of the slow variable x_2. In particular, we must make the following replacement:

$$\frac{\partial F(x,t)}{\partial x} \rightarrow \frac{\partial F(x,x_2,t)}{\partial x} + \frac{\partial F(x,x_2,t)}{\partial x_2}\frac{\partial x_2}{\partial x} = \frac{\partial F}{\partial x} + \epsilon^2 \frac{\partial F}{\partial x_2} \tag{1.3}$$

The following perturbation equations result:

$O(\epsilon^0)$:

$$\rho \frac{\partial^2 V_0}{\partial t^2} - T\frac{\partial^2 V_0}{\partial x^2} + KV_0 = 0 \tag{1.4}$$

$O(\epsilon)$:

$$\rho \frac{\partial^2 V_1}{\partial t^2} - T\frac{\partial^2 V_1}{\partial x^2} + KV_1 + KMV_0 = 0 \tag{1.5}$$

$O(\epsilon^2)$:

$$\rho \frac{\partial V_2}{\partial t^2} - T\frac{\partial^2 V_2}{\partial x^2} + KV_2 + KMV_1 - T\left(2\frac{\partial^2 V_0}{\partial x \partial x_2}\right) = 0. \tag{1.6}$$

Let us take the leading-order solution to be a progressive wave

$$V_0 = A\left(x_2\right) e^{i(kx-\omega t)} \tag{1.7}$$

with the dispersion relation

$$\omega = \left(\frac{Tk^2 + K}{\rho}\right)^{1/2}, \quad C = \frac{\omega}{k} = \left(\frac{T}{\rho} + \frac{K}{\rho k^2}\right)^{1/2}. \tag{1.8}$$

At the order $O(\epsilon)$ Eqn. (1.5) can be written

$$\rho \frac{\partial^2 V_1}{\partial t^2} - T\frac{\partial^2 V_1}{\partial x^2} + KV_1 = -KM(x,x_2)Ae^{ikx-i\omega t}. \tag{1.9}$$

where the forcing term on the right is a random function of x and x_2. Let

$$V_1 = \overline{V}_1 e^{-i\omega t} \tag{1.10}$$

then

$$\frac{\partial^2 \overline{V}_1}{\partial x^2} + k^2 \overline{V}_1 = \frac{K}{T} M(x,x_2)A(x_2)e^{ikx}, \tag{1.11}$$

where use is made of

$$k = \left(\frac{\rho\omega^2 - K}{T}\right)^{1/2}. \tag{1.12}$$

From here on we consider the frequency to be above cutoff $\sqrt{K/T}$ so that k is real and positive. Equation (1.11) can be solved by using the Green function,

$$G = \frac{-i}{2k} e^{ik|x-\xi|} \tag{1.13}$$

which satisfies the radiation condition at infinities. The result is

$$\overline{V}_1 = \frac{-iKA}{2kT} \int_{-\infty}^{\infty} e^{ik|x-\xi|} M(x) e^{ik\xi} d\xi \tag{1.14}$$

so that

$$V_1 = \frac{-iKA}{2kT} e^{ikx-i\omega t} \int_{-\infty}^{\infty} d\xi e^{ik|x-\xi|} e^{-ik(x-\xi)} M(x) \tag{1.15}$$

which behaves as outgoing waves as $x - \xi \to \pm\infty$.

Since the ensemble average of $M(x)$ vanishes, i.e., $\langle M(x) \rangle = 0$, we have,

$$\langle V_1 \rangle = \langle \overline{V}_1 \rangle e^{-i\omega t} = 0. \tag{1.16}$$

The ensemble average of Eq. (1.6) becomes

$$\rho \frac{\partial^2 \langle V_2 \rangle}{\partial t^2} - T \frac{\partial^2 \langle V_2 \rangle}{\partial x^2} + K \langle V_2 \rangle - T \left(2ik \frac{\partial V_0}{\partial x_2} \right) + \left[K \left\{ \frac{-iK}{2kT} \int e^{ik|x-\xi|} \langle M(x) M(\xi) \rangle e^{-ik(x-\xi)} d\xi \right\} e^{ikx-i\omega t} \right] = 0 \tag{1.17}$$

where $\langle M(x) M(\xi) \rangle$ is the correlation function of the irregularities. Equating the sum of all secular terms to zero, we get the evolution equation for A,

$$-2i\omega\rho \left(C_g \frac{\partial A}{\partial x_2} \right) - i \frac{K^2 A}{2kT} \int_{-\infty}^{\infty} \langle M(x) M(\xi) \rangle e^{ik|x-\xi|} e^{-k(x-\xi)} d\xi = 0 \tag{1.18}$$

where where

$$C_g = \frac{\partial \omega}{\partial k} = \frac{Tk}{\rho\omega} \tag{1.19}$$

is the group velocity.

As a specific example we take

$$\langle M(x) M(\xi) \rangle = \sigma^2 e^{-\alpha|x-\xi|}, \tag{1.20}$$

thus $\epsilon\sigma$ corresponds to the root-mean-square of the fluctuation amplitude. The corrleation length scale is $1/\alpha$. small α means a high degree of randomness. It can be shown that

$$\int_{-\infty}^{\infty} e^{ik|x-\xi|} \sigma^2 e^{-\alpha|x-\xi|} e^{-ik(x-\xi)} d\xi = \sigma^2 \left[\frac{2\left(\alpha^2 + 2k^2\right)}{\alpha(\alpha^2 + 4k^2)} + \frac{2ik}{\alpha^2 + 4k^2} \right]. \tag{1.21}$$

(Chen & Soong, 1972). Let

$$2\beta = 2\left(\beta_r + i\beta_i\right) \tag{1.22}$$

with

$$\beta_r = -\left(\frac{K^2\sigma^2}{2\rho\omega T}\right)\frac{1}{\alpha^2+4k^2}, \quad \beta_i = \left(\frac{K^2\sigma^2}{2\rho\omega T}\right)\frac{(\alpha^2+2k^2)}{k\alpha(\alpha^2+4k^2)}. \tag{1.23}$$

we get from (1.17)

$$2i\left(C_g\frac{\partial A}{\partial x_2}\right) + 2\beta A = 0. \tag{1.24}$$

If we write

$$A = a(x_2)e^{i\theta(x_2)} \tag{1.25}$$

where a is the magnitude and θ the phase of A. From the real part we get

$$\theta = \beta_r x_2/C_g \tag{1.26}$$

From the imaginary part we get

$$2C_g\frac{\partial a}{\partial x_2} + 2\beta_i a = 0, \tag{1.27}$$

hence

$$a(x_2) = A_0 \exp\left(-\frac{\beta_i x_2}{C_g}\right). \tag{1.28}$$

showing that the incident waves are attenuated (localized) by random irregularities.The ratio of attenuation (localization) distance is

$$L = \frac{C_g}{\epsilon^2\beta_i} = \frac{2\rho\omega C_g T k}{\epsilon^2 K^2\sigma^2}\frac{\alpha(\alpha^2+4k^2)}{\alpha^2+2k^2} = \frac{2T^2k^2}{\epsilon^2K^2\sigma^2}\frac{\alpha(\alpha^2+4k^2)}{\alpha^2+2k^2}. \tag{1.29}$$

For fixed α and k, L is small (strong attenuation) if the fluctuation amplitude $\epsilon\sigma$ is large. For fixed $\epsilon\sigma$, L is also large for large k (short waves) or large α, corresponding to small correlation distance (very random).

The total change in wave number due to randomness is

$$\Delta k = \frac{\epsilon^2\beta_r}{C_g} = -\frac{\epsilon^2}{C_g}\frac{K^2\sigma^2}{2\rho\omega kT}\frac{1}{\alpha^2+4k^2} = -\frac{\epsilon^2K^2\sigma^2}{2T^2k}\frac{2k}{\alpha}\frac{1}{\alpha^2+4k^2} \tag{1.30}$$

It is negative, hences contributes to the lengthening of waves or increase in phase velocity. The magnitude of the wavenumber shift increases with increasing $\epsilon^2\sigma^2$ and descreasing α (decreasing randomness).

IAP (challenge) Project : Scattering of elastic waves by random distribution of hard grains or cavities.

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In short, physics has discovered

That there are no solids,

No continuous surfaces,

No straight lines;

Only waves,

....

R. Buckminster Fuller, *Intuition: Metaphysical Mosaic.*